Available online at www.sciencedirect.com
ScienceDirect
Mathematics and Computers in Simulation 79 (2009) 1527–1543
MATHEMATICS AND COMPUTERS IN SIMULATION
www.elsevier.com/locate/matcom

New criteria for globally exponential stability of delayed Cohen–Grossberg neural network

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Received 3 October 2006; received in revised form 16 April 2007; accepted 1 July 2008
Available online 23 July 2008

Abstract

This paper is concerned with analysis problem for the global exponential stability of the Cohen–Grossberg neural networks with discrete delays and with distributed delays. We first prove the existence and uniqueness of the equilibrium point under mild conditions, assuming neither differentiability nor strict monotonicity for the activation function. Then, we employ Lyapunov functions to establish some sufficient conditions ensuring global exponential stability of equilibria for the Cohen–Grossberg neural networks with discrete delays and with distributed delays. Our results are not only presented in terms of system parameters and can be easily verified and also less restrictive than previously known criteria. A comparison between our results and the previous results admits that our results establish a new set of stability criteria for delayed neural networks.

Keywords: Global exponential stability (GES); Lyapunov functional; Neural networks; Delays

1. Introduction

Recently, Cohen and Grossberg [5] proposed and studied an artificial feedback neural network, which is described by a system of ordinary differential equations.

$$\dot{x}_i(t) = -a_i(x_i(t))\left[b_i(x_i(t)) - \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) + J_i\right], \quad i = 1, 2, \ldots, n, \tag{1.1}$$

where n is the number of neurons in the network; x_i describes the activation of the ith neuron; a_i represents an amplification function and the function b_i can be include a constant term indicating a fixed input to the network; the $n \times n$ connection matrix $A = (a_{ij})$ tells how the neurons are connected in the network; the activation functions f_j, $j = 1, 2, \ldots, n$, show how the neurons react to input, and J_i, $i = 1, 2, \ldots, n$, denote the constant inputs from outside of the system.

Due to its promising potential for the tasks of classification, associative memory, parallel computations, and its ability to solve difficult optimization problems, (1.1) has greatly attracted the attention of the scientific community. Various

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doi:10.1016/j.matcom.2008.07.002

generalizations and modifications of (1.1) have also been proposed and studied, among which is the incorporation of time delay into the model. In fact, due to the finite speeds of the switching and transmission of signals in a network, time delays do exist in a working network and, thus, should be incorporated into the model equations of the network. For more detailed justifications for introducing delays into model equations of neural networks [1–4,13,14]. For the Cohen–Grossberg model (1.1), Ye et al. [21] first introduced delays by considering the following system of delay differential equations:

$$\dot{x}_i(t) = -a_i(x_i(t))\left[b_i(x_i(t)) - \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) - \sum_{j=1}^{n} b_{ij} f_j(x_j(t-\tau_{ij}(t))) + J_i\right], \quad i = 1, 2, \ldots, n, \tag{1.2}$$

where $a_i(x_i(t)) > 0$, $b_i(x_i(t))$ are called the amplification and the self-signal functions, respectively. $A = (a_{ij})_{n\times n}$ and $B = (b_{ij})_{n\times n}$ are the normal and the delayed connection weight matrix, respectively. Moreover, $f_j(\cdot)$ denote the neuron activation functions. The variable delays $\tau_{ij}(t)$, $i, j = 1, 2, \ldots, n$, are continuous and differential with $0 \le \tau_{ij}(t) \le \tau$ and $\dot{\tau}_{ij}(t) \le \eta < 1$ for nonnegative constant η (If $\dot{\tau}_{ij}(t) \le 0$, let $\eta = 0$). We also note that, some authors [8,13,19] have studied the pure-delay model (with $a_{ij} = 0$, $i, j, = 1, 2, \ldots, n$), and here we consider the above hybrid model in which both instantaneous and delayed signaling occur (with $a_{ij} \ne 0$ and $b_{ij} \ne 0$).

Although the use of constant fixed delays in models of delayed neural networks provides a good approximation in simple circuits composed of a small number of cells, neural networks usually have a spatial extent due to the presence of a multitude of parallel pathways with a variety of axon sizes and lengths. Thus, there will be distribution of propagation delays. In these circumstances, the signal propagation is not instantaneous and cannot modeled with discrete delays. A more appropriate way is to incorporate distributed delays. However, the criteria for stability are mostly for systems with discrete delays. Only a few of them are for neural networks with distributed delays; see, for example, [8,14,17,20,22,23]. However, those authors have only studied the pure-delay model (with $a_{ij} = 0$, $i, j, = 1, 2, \ldots, n$). Hence, in this paper, we also consider the following Cohen–Grossberg hybrid models with distributed delays:

$$\dot{x}_i(t) = -a_i(x_i(t))\left[b_i(x_i(t)) - \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) - \sum_{j=1}^{n} b_{ij} \int_{-\infty}^{t} K_{ij}(t-s) f_j(x_j(s))\,\mathrm{d}s + J_i\right], \quad i = 1, 2, \ldots, n, \tag{1.3}$$

where the definitions of a_i, b_i, a_{ij} and b_{ij} are the same as those for systems (1.2). The delay kernel functions $k_{ij}(t)$ are assumed to be real valued nonnegative and piecewise continuous functions defined on $[0, +\infty)$, and satisfy

$$\int_0^{+\infty} k_{ij}(t)\,\mathrm{d}t = 1. \tag{1.4}$$

One of the most investigated problems is that of the existence, uniqueness, global asymptotic stability (GAS), and global exponential stability (GES) of the equilibrium. The number of equilibria of the neural network relates to its storage capacity. When designing an associative memory neural network, we should make as many stable equilibrium states as possible to provide a memory system with large information capability, an attractive region of each stable equilibrium state as large as possible to provide the robustness and fault tolerance for information processing, and a convergence speed as high as possible to ensure the quick convergence of the network operation. Due to the properties of locally asymptotic stability, the associative memory network is used mainly for information retrieval, pattern recognition, etc. On the other hand, to embed and solve many problems in applications of neural networks to parallel computations, signal processing and other problems involving the optimization, the dynamic neural networks have to be designed to have only a unique equilibrium point which is GAS or GES to avoid the risk of spurious responses or the problem of local minima. In fact, earlier applications of neural networks to optimization problems have suffered from the existence of a complicated set of equilibria (see [18]). Thus, the GAS and GES of a unique equilibrium for the model system is of great importance from a theoretical and an application point of view in several fields, and has been the major concern of many authors. Thus, the primary purpose in this paper is to obtain some criteria ensuring that (1.2) and (1.3) have a unique equilibrium which is globally exponentially stable (GES). Some existing results on existence, uniqueness, GAS, and GES of the equilibrium concern the case where the activation functions are all bounded and strictly increasing. These assumptions make the results inapplicable to some important engineering problems. However, in this paper, not only do we abandon the boundedness condition of f_i, but also we remove the differentiability and strict monotonicity restriction from f_i, although this will lead to move difficulty in stability analysis. In addition, we do not impose any restriction such as symmetry on the connection matrix.

The rest of this paper is organized as follows. In the next Section, some notations and assumptions are given, and then in Section 3, Existence and uniqueness of the equilibrium are addressed by employing homeomorphism techniques. By constructing novel Lyapunov functionals, we studied the Cohen–Grossberg models with discrete delay and with distributed delays. Some exponential stability criteria for the various networks are presented in Sections 4 and 5, respectively. Examples and comparisons are given in Section 6 to demonstrate the effectiveness of our main results. Finally, conclusions are drawn in Section 7.

2. Preliminaries

Firstly, throughout this paper we will use the following notations: Let $B = (b_{ij})$ be a real matrix of dimension of $n \times n$. B^{T}, B^{-1} denotes, respectively, the transpose of, the inverse of a square matrix B. The notation $B > 0(B < 0)$ means that B is symmetric and positive definite (negative definite). $\|B\|_2$ represent the norm of B induced by the Euclidean vector norm, i.e., $\|B\|_2 = (\lambda(B^{\mathrm{T}}B))^{1/2}$, where $\lambda(M)$ represents the maximum eigenvalue of matrix M. $\|B\|_1$ and $\|B\|_\infty$ represent the first norm and infinity norm of B, respectively. That is, $\|B\|_1 = \max_{1\le j\le n}\sum_{i=1}^{n}|b_{ij}|$, $\|B\|_\infty = \max_{1\le i\le n}\sum_{j=1}^{n}|b_{ij}|$. B_1 and B_∞ represent diagonal matrices $\mathrm{diag}\left(\sum_{i=1}^{n}|b_{i1}|, \sum_{i=1}^{n}|b_{i2}|, \ldots, \sum_{i=1}^{n}|b_{in}|\right)$ and $\mathrm{diag}\left(\sum_{i=1}^{n}|b_{1i}|, \sum_{i=1}^{n}|b_{2i}|, \ldots, \sum_{i=1}^{n}|b_{ni}|\right)$, respectively. Secondly, in order to establish the global stability conditions for the above neural networks and make a precise comparison between our stability conditions and previous results derived in the literature, we first give some usual assumptions on the functions a_i, k_i, b_i and f_i:

Assumption A_1. The functions $a_i(x)$, $i = 1, 2, \ldots, n$, are continuously bounded, and there exist positive constants $\underline{\alpha}_i$ and $\bar{\alpha}_i$ such that $0 < \underline{\alpha}_i \le a_i(x) \le \bar{\alpha}_i, \quad \forall x \in \mathbb{R}$.

Assumption A_2. The functions $b_i(x)$ are continuous and there exist constants $\gamma_i > 0$ such that

$$(b_i(x) - b_i(y))(x - y) \ge \gamma_i(x - y)^2 > 0, \; i = 1, 2, \ldots, n, \quad \forall x, y \in \mathbb{R}, x \ne y.$$

Assumption A_3. There exist some positive constants G_i such that

$$0 \le (f_i(x) - f_i(y))(x - y) \le G_i(x - y)^2, \; i = 1, 2, \ldots, n, \quad \forall x, y \in \mathbb{R}, x \ne y.$$

Assumption A_4. There exists a positive number δ_0 such that $\int_0^\infty k_{ij}(s)\mathrm{e}^{\delta_0 s}\,\mathrm{d}s < \infty$.

3. Existence and uniqueness of equilibrium

In this section, we prove the existence and the uniqueness of the equilibrium point of the Cohen–Grossberg networks under very relaxed conditions. Before the proof, let us review a lemma given in [7]

Lemma 3.1. *Continuous map $H(x) : R^n \to R^n$ is homeomorphic if $H(x)$ is injective and* $\lim_{\|x\|\to\infty}\|H(x)\| = \infty$.

We also have the following lemma due to [16]

Lemma 3.2. *Given any real matrices X, Y, C of appropriate dimensions and a scalar $\epsilon_0 > 0$, where $C > 0$. Then $X^{\mathrm{T}}Y + Y^{\mathrm{T}}X \le \epsilon_0 X^{\mathrm{T}}CX + (1/\epsilon_0)Y^{\mathrm{T}}C^{-1}Y$.*

Since $a_i(x)$ is positive, a point $x^* = (x_1^*, x_2^x, \ldots, x_n^*)^{\mathrm{T}}$ is an equilibrium of system (1.2)(or (1.3)) if and only if this point x^* is a solution of the following equation:

$$b_i(x_i) - \sum_{j=1}^{n}(a_{ij} + b_{ij})f_j(x_j) + J_i = 0, \quad i = 1, 2, \ldots, n. \tag{3.1}$$

Generally, (3.1) may have more than one solution x^* and, hence, system (1.2)(or (1.3)) maybe have more than one equilibrium. It is well known that bounded activation functions always guarantee the existence of an equilibrium point for system (1.2). Moreover, we have the following theorem for unbounded activation functions:

Theorem 3.3. *Suppose that in systems (1.2)(or (1.3)),* Assumptions A_1– A_3 *are satisfied, and* $|f_j(x)| \to \infty$ *as* $|x| \to \infty$. *Neural system (1.2)(or (1.3)) has a unique equilibrium point if there exist symmetric positive diagonal matrices* $P = diag(p_1, p_2, \ldots, p_n)$, *and* $Q = diag(q_1, q_2, \ldots, q_n)$ *satisfying that*:

$$\Omega_1 = 2P\Gamma G^{-1} - PA - A^{\mathrm{T}}P - (PQ^{-1}B)_\infty - (PQB)_1 > 0$$

where $\Gamma = diag(\gamma_1, \gamma_2, \ldots, \gamma_n)$.

Proof. Let $x^* = (x_1^*, x_2^*, \ldots, x_n^*)^{\mathrm{T}}$ denotes an equilibrium point of neural network model (1.2). Then, x^* satisfies (3.1). Let

$$F(x) = (F_1(x), F_2(x), \ldots, F_n(x))^{\mathrm{T}} = 0. \tag{3.2}$$

where $F_i(x) = b_i(x_i) - \sum_{j=1}^{n}(a_{ij} + b_{ij})f_j(x_j) + J_i$.

Obviously, the solution of (3.2) is the equilibrium point of (1.2). Therefore, (1.2) has a unique equilibrium point if $F(x)$ is homeomorphism of $\mathbb{R}^n$. From Lemma 3.1, we know that $F(x)$ is homeomorphism of $\mathbb{R}^n$ if $F(x) \neq F(y), \forall x \neq y, x, y, \in \mathbb{R}^n$, and $\|F(x)\| \to \infty$ as $\|x\| \to \infty$. Let x and y be two vectors such $x \neq y$. Under the assumptions on the activation functions, $x \neq y$ will imply two cases: (i) $x \neq y$ and $f(x) - f(y) \neq 0$, (ii) $x \neq y$ and $f(x) - f(y) = 0$, where $f(x) = (f_1(x_1), f_2(x_2), \ldots, f_n(x_n))^{\mathrm{T}}$.

First, consider the case where $x \neq y$ and $f(x) - f(y) \neq 0$. In this case, for $P = \mathrm{diag}(p_1, p_2, \ldots, p_n)$, multiplying both sides of the first equation in (3.2) by $2(f(x) - f(y))^{\mathrm{T}}P$ results in

$$2(f(x) - f(y))^{\mathrm{T}}P(F(x) - F(y)) = -2(f(x) - f(y))^{\mathrm{T}}P[\beta(x) - \beta(y) - (A + B)(f(x) - f(y))] \tag{3.3}$$

where $\beta(x) = (b_1(x_1), b_2(x_2), \ldots, b_n(x_n))^{\mathrm{T}}$.

Since $(b_i(x_i) - b_i(y_i))(f_i(x_i) - f_i(y_i)) \geq (\gamma_i/G_i)(f_i(x_i) - f_i(y_i))^2$, we have

$$(f(x) - f(y))^{\mathrm{T}}P(\beta(x) - \beta(y)) \geq (f(x) - f(y))^{\mathrm{T}}P\Gamma G^{-1}(f(x) - f(y)).$$

Since $2b_{ij}(f_i(x_i) - f_i(y_i))(f_j(x_j) - f_j(y_j)) \leq q_i^{-1}|b_{ij}|(f_i(x_i) - f_i(y_i))^2 + q_i|b_{ij}|(f_j(x_j) - f_j(y_j))^2$, we have

$$\begin{aligned} 2(f(x) - f(y))^{\mathrm{T}}P(F(x) - F(y)) &\leq -2(f(x) - f(y))^{\mathrm{T}}P\Gamma G^{-1}(f(x) - f(y)) \\ &+(f(x) - f(y))^{\mathrm{T}}(PA + A^{\mathrm{T}}P)(f(x) - f(y)) + (f(x) - f(y))^{\mathrm{T}}((PQ^{-1}B)_\infty \\ &+(PQB)_1)(f(x) - f(y)) \leq -(f(x) - f(y))^{\mathrm{T}}\Omega_1(f(x) - f(y)) < 0 \end{aligned} \tag{3.4}$$

Thus, $F(x) \neq F(y)$ when $f(x) \neq f(y)$ as P is a positive diagonal matrix. Hence, we have proved that $F(x) - F(y) \neq 0$ when $x \neq y$ and $f(x) \neq f(y)$. Now consider the case (ii) where $x \neq y$ and $f(x) - f(y) = 0$. In the case, we have

$$F(x) - F(y) = -(\beta(x) - \beta(y)) \neq 0,$$

thus, implying that $F(x) \neq F(y)$ when $x \neq y$ and $f(x) = f(y)$. Hence, we have proved that $F(x) \neq F(y)$ for all $x \neq y$.

We will show that the conditions of Theorem 3.3 also imply that $\|F(x)\| \to \infty$ as $\|x\| \to \infty$. To this end, in (3.4), let $y = 0$, which yields

$$2(f(x) - f(0))^{\mathrm{T}}P(F(x) - F(0)) \leq -(f(x) - f(0))^{\mathrm{T}}\Omega_1(f(x) - f(0)) \leq -\lambda(f(x) - f(0))^{\mathrm{T}}(f(x) - f(0)) \tag{3.5}$$

where λ is the minimum eigenvalue of the positive-definite matrix Ω_1. From (3.5), it follows that

$$\begin{aligned} \lambda\|f(x) - f(0)\|_2^2 &\leq \left|\sum_{i=1}^{n}2p_i(f_i(x_i) - f_i(0))(F_i(x) - F_i(0))\right| \leq 2p\sum_{i=1}^{n}|f_i(x_i) - f_i(0)||F_i(x) - F_i(0)| \leq 2p\|f(x) \\ &- f(0)\|_\infty\sum_{i=1}^{n}|F_i(x) - F_i(0)| = 2p\|f(x) - f(0)\|_\infty\|F(x) - F(0)\|_1, \end{aligned}$$

where $p = \max\{p_1, p_2, \ldots, p_n\}$. Using the fact $\|f(x) - f(0)\|_\infty \leq \|f(x) - f(0)\|_2$, we obtain

$$\lambda\|f(x) - f(0)\|_\infty \leq 2p\|F(x) - F(0)\|_1.$$

We note that $\|f(x) - f(0)\|_\infty \geq \|f(x)\|_\infty - \|f(0)\|_\infty$ and $\|F(x) - F(0)\|_1 \leq \|F(x)\|_1 + \|F(0)\|_1$. Thus, we have

$$\lambda\|f(x)\|_\infty - \lambda\|f(0)\|_\infty \leq 2p\|F(x)\|_1 + 2p\|F(0)\|_1.$$

Hence,

$$\|F(x)\|_1 \geq \frac{\lambda\|f(x)\|_\infty - \lambda\|f(0)\|_\infty - 2p\|F(0)\|_1}{2p}$$

from which it can be easily concluded that $\|F(x)\| \to \infty$ as $\|f(x)\| \to \infty$. From that $|f_j(x)| \to \infty$ as $|x| \to \infty$, we can obtain that $\|F(x)\| \to \infty$ as $\|x\| \to \infty$. Hence, we have proved that $F(x)$ is homeomorphism of $\mathbb{R}^n$, thus implying that the neural system (1.2)(or (1.3)) has an equilibrium point and this equilibrium point is unique. □

Similar to that of Theorem 3.3, we can obtain the following theorems.

Theorem 3.4. *Suppose that in systems (1.2)(or (1.3)),* Assumptions A_1– A_3 *are satisfied, and* $|f_j(x)| \to \infty$ *as* $|x| \to \infty$. *Neural system (1.2)(or (1.3)) has a unique equilibrium point if there exist symmetric positive diagonal matrices* $P = diag(p_1, p_2, \ldots, p_n)$, *and* $Q = diag(q_1, q_2, \ldots, q_n)$ *such that:*

$$\Omega_2 = 2P\Gamma G^{-1} - PA - A^{\mathrm{T}}P - (PBQ^{-1})_\infty - (PBQ)_1 > 0$$

where $\Gamma = diag(\gamma_1, \gamma_2, \ldots, \gamma_n)$.

Theorem 3.5. *Suppose that in systems (1.2)(or (1.3)),* Assumptions A_1– A_3 *are satisfied, and* $|f_j(x)| \to \infty$ *as* $|x| \to \infty$. *Neural system (1.2)(or (1.3)) has a unique equilibrium point if there exist symmetric positive diagonal matrices* $P = diag(p_1, p_2, \ldots, p_n)$, *and* $Q = diag(q_1, q_2, \ldots, q_n)$ *such that:*

$$\Omega_3 = 2P\Gamma G^{-1} - PA - A^{\mathrm{T}}P - nPQ^{-1} - PQdiag\left(\sum_{j=1}^{n} b_{j1}^2, \sum_{j=1}^{n} b_{j2}^2, \ldots, \sum_{j=1}^{n} b_{jn}^2\right) > 0,$$

where $\Gamma = diag(\gamma_1, \gamma_2, \ldots, \gamma_n)$.

4. The Cohen–Grossberg model with discrete delays

In this section, we consider system (1.2). Its initial conditions are of the form:

$$x_i(s) = \varphi_i(s) \in C([-\tau, 0], R), s \in [-\tau, 0], i = 1, 2, \ldots, n, \tag{4.1}$$

where $\tau = \max_{1 \leq i,j \leq n} \tau_{ij}$.

Let x^* be an equilibrium of system (1.2) and $z(t) = x(t) - x^*$. Substituting $x(t) = z(t) + x^*$ into system (1.2) leads to

$$\dot{z}_i(t) = -\alpha_i(z_i(t)) \left[\beta_i(z_i(t)) - \sum_{j=1}^{n} a_{ij} g_j(z_j(t)) - \sum_{j=1}^{n} b_{ij} g_j(z_j(t - \tau_{ij}(t)))\right], \tag{4.2}$$

where

$$\alpha_i(z_i(t)) = a_i(z_i(t) + x_i^*), \beta_i(z_i(t)) = b_i(z_i(t) + x_i^*) - b_i(x_i^*) g_i(z_i(t)) = f_i(z_i(t) + x_i^*) - f_i(x_i^*). \tag{4.3}$$

With respect to the exponential stability of x^* for the system (1.2), we have the following result which is independent of delays.

Theorem 4.1. *Suppose that in systems (4.2),* $\dot{\tau}_{ij}(t) \leq \eta < 1$*for non-negative constant* η *(If* $\dot{\tau}_{ij}(t) \leq \eta \leq 0$, *let* $\eta = 0$), $0 \leq \tau(t) \leq \tau$, Assumptions A_1– A_3 *are satisfied. If there exist symmetric positive diagonal matrices* $P =$

diag$(p_1, p_2, \ldots, p_n)$, *and* $Q = diag(q_1, q_2, \ldots, q_n)$ *such that:*

$$\Omega_{11} = 2P\Gamma G^{-1} - PA - A^{\mathrm{T}}P - (PQ^{-1}B)_{\infty} - \frac{1}{1-\eta}(PQB)_1 > 0,$$

then the origin of neural system (4.2) is exponentially stable. This implies that there exist positive constants k and γ *such that for any solution* $z(t)$ *of system (4.2) with initial function* $z(s) = \phi(s)$ *for all* $s \in [-\tau, 0]$, *where* $\phi \in C([-\tau, 0], \mathbb{R}^n)$, *one has*

$$\|z(t)\|_2^2 \equiv \sum_{i=1}^{n} \sup_{t\in[-\tau,0]} z_i^2(t) \leq \gamma e^{-kt} \sum_{i=1}^{n} \sup_{s\in[-\tau,0]} \phi_i^2(s) = \gamma e^{-kt} \|\phi\|_2^2.$$

Proof. Since $\Omega_1 \geq \Omega_{11} > 0$ and Lemma 3.3, the origin of neural system (4.2) is a unique equilibrium point.

We employ the following positive-definite Lyapunov functional:

$$V(z(t), t) = \epsilon_1 V_1(z(t), t) + V_2(z(t), t), \tag{4.4}$$

where

$$V_1(z(t), t) = 2\sum_{i=1}^{n} \int_0^{z_i(t)} \frac{s}{\alpha_i(s)}\,ds,$$

$$V_2(z(t), t) = 2\sum_{i=1}^{n} p_i \int_0^{z_i(t)} \frac{g_i(s)}{\alpha_i(s)}\,ds + \frac{1}{1-\eta}\sum_{i=1}^{n}\sum_{j=1}^{n} \int_{t-\tau_{ji}(t)}^{t} r_j|b_{ji}|g_i^2(z_i(v))\,dv,$$

for positive constant ϵ_1 and r_i, $i = 1, 2, \ldots, n$. The positive constant ϵ_1 and r_i will be determined later. The derivative of V along trajectories of (4.2) is given by:

$$\dot{V}(z(t), t) = \epsilon_1 \dot{V}_1(z(t), t) + \dot{V}_2(z(t), t),$$

where

$$\dot{V}_1(z(t), t) = 2\sum_{i=1}^{n} z_i(t)\left[-\beta_i(z_i(t)) + \sum_{j=1}^{n} a_{ij} g_j(z_j(t)) + \sum_{j=1}^{n} b_{ij} g_j(z_j(t - \tau_{ij}(t)))\right]$$

and

$$\dot{V}_2(z(t), t) = 2\sum_{i=1}^{n} g_i(z_i(t)) p_i \left[-\beta_i(z_i(t)) + \sum_{j=1}^{n} a_{ij} g_j(z_j(t)) + \sum_{j=1}^{n} b_{ij} g_j(z_j(t - \tau_{ij}(t)))\right]$$
$$+ \frac{1}{1-\eta}\sum_{i=1}^{n}\sum_{j=1}^{n} r_j|b_{ji}|g_i^2(z_i(t)) - \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{1-\dot{\tau}_{ji}(t)}{1-\eta} r_j|b_{ji}|g_i^2(z_i(t - \tau_{ji}(t)))$$

Since $z_i(t)\beta_i(z_i(t)) \geq \gamma_i z_i^2(t)$, we can rewrite $\dot{V}_1$ as the following format:

$$\dot{V}_1(z(t), t) \leq -2\sum_{i=1}^{n} \gamma_i z_i^2(t) + 2\left(z^{\mathrm{T}}(t)\frac{\Gamma^{1/2}}{\sqrt{2}}\right)(\sqrt{2}\Gamma^{-(1/2)} A g(z(t))) + \sum_{i=1}^{n}\sum_{j=1}^{n} 2 z_i(t) b_{ij} g_j(z_j(t - \tau_{ij}(t))),$$

where $g(z(t)) = (g_1(z_1(t)), g_2(z_2(t)), \ldots, g_n(z_n(t)))^{\mathrm{T}}$.

By Lemma 3.2 and the Cauchy inequality (i.e. $a^2 + b^2 \geq 2ab$), we get

$$\dot{V}_1(z(t), t) \leq -z^{\mathrm{T}}(t)\Gamma z(t) + 2g(z(t))^{\mathrm{T}} A^{\mathrm{T}} \Gamma^{-1} A g(z(t)) + \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{2}{\gamma_i} b_{ij}^2 g_j^2(z_j(t - \tau_{ij}(t))) \leq -z^{\mathrm{T}}(t)\Gamma z(t)$$

$$+ M g(z(t))^{\mathrm{T}} g(z(t)) + M\sum_{i=1}^{n}\sum_{j=1}^{n} |b_{ij}| g_j^2(z_j(t - \tau_{ij}(t)))$$

where $M = \max\{2\|A^{\mathrm{T}}\Gamma^{-1}A\|_2, \max_{1\le i,j\le n}(2/\gamma_i)|b_{ij}|\} \ge 0$.

Since $g_i(z_i(t))\beta_i(z_i(t)) \ge \gamma_i z_i(t) g_i(z_i(t)) \ge \gamma_i G_i^{-1}(g_i(z_i(t)))^2$, we have

$$-g(z(t))^{\mathrm{T}}P\beta(z(t)) \le -g(z(t))^{\mathrm{T}}P\Gamma z(t) \le -g(z(t))^{\mathrm{T}}P\Gamma G^{-1}g(z(t)).$$

And since $((1-\dot{\tau}_{ij}(t))/(1-\eta)) \le 1$, the term $\dot{V}_2$ can be bounded.

$$\dot{V}_2(z(t),t) \le -g(z(t))^{\mathrm{T}}\left(2P\Gamma G^{-1} - PA - A^{\mathrm{T}}P - \frac{1}{1-\eta}(RB)_1\right)g(z(t)) + 2\sum_{i=1}^{n}p_i g_i(z_i(t))\sum_{j=1}^{n}b_{ij}g_j(z_j(t-\tau_{ij}(t))) - \sum_{i=1}^{n}\sum_{j=1}^{n}r_j|b_{ji}|g_i^2(z_i(t-\tau_{ji}(t)))$$

Since $g_i(z_i(t))b_{ij}g_j(z_j(t-\tau_{ij}(t))) \le ((|b_{ij}|)/q_i)g_i^2(z_i(t)) + q_i|b_{ij}|g_j^2(z_j(t-\tau_{ij}(t)))$, we have

$$\dot{V}_2(z(t),t) \le -g(z(t))^{\mathrm{T}}(2P\Gamma G^{-1} - PA - A^{\mathrm{T}}P - \frac{1}{1-\eta}(RB)_1)g(z(t)) + \sum_{i=1}^{n}\sum_{j=1}^{n}p_i q_i^{-1}|b_{ij}|g_i^2(z_i(t)) + \sum_{i=1}^{n}\sum_{j=1}^{n}p_i q_i|b_{ij}|g_j^2(z_j(t-\tau_{ij}(t))) - \sum_{i=1}^{n}\sum_{j=1}^{n}r_j|b_{ji}|g_i^2(z_i(t-\tau_{ji}(t))) \le -g(z(t))^{\mathrm{T}} \left(2P\Gamma G^{-1} - PA - A^{\mathrm{T}}P - \frac{1}{1-\eta}(RB)_1\right)g(z(t)) + \sum_{i=1}^{n}\sum_{j=1}^{n}p_i q_i^{-1}|b_{ij}|g_i^2(z_i(t)) - \sum_{i=1}^{n}\sum_{j=1}^{n}(r_i - p_i q_i)|b_{ij}|g_j^2(z_j(t-\tau_{ij}(t)))$$

Since $\Omega_{11} > 0$, there exists $\epsilon_2 > 0$ such that $\Omega_{11} - \epsilon_2(I + (1/1-\eta)B_1) > 0$. If we define $r_i = p_i q_i + \epsilon_2$, we can have

$$\dot{V}_2(z(t),t) \le -g(z(t))^{\mathrm{T}}\left(2P\Gamma G^{-1} - PA - A^{\mathrm{T}}P - (PQ^{-1}B)_\infty - \frac{\epsilon_2}{1-\eta}B_1 - \frac{1}{1-\eta}(PQB)_1\right)g(z(t)) - \epsilon_2\sum_{i=1}^{n}\sum_{j=1}^{n}|b_{ij}|g_j^2(z_j(t-\tau_{ij}(t))) \le -g(z(t))^{\mathrm{T}}\left[\left(\Omega_{11} - \epsilon_2\left(I + \frac{1}{1-\eta}B_1\right)\right) + \epsilon_2 I\right]g(z(t)) - \epsilon_2\sum_{i=1}^{n}\sum_{j=1}^{n}|b_{ij}|g_j^2(z_j(t-\tau_{ij}(t))) \le -\epsilon_2 g(z(t))^{\mathrm{T}}g(z(t)) - \epsilon_2\sum_{i=1}^{n}\sum_{j=1}^{n}|b_{ij}|g_j^2(z_j(t-\tau_{ij}(t))).$$

If we choose $\epsilon_1 > 0$ such that $M\epsilon_1 \le \epsilon_2$, then $\dot{V}(z(t),t) \le -\epsilon_1 z(t)^{\mathrm{T}}\Gamma z(t)$.

Let $\gamma = \min_{1\le i\le n}\gamma_i$, $r = \max_{1\le i\le n}\sum_{j=1}^{n}r_j|b_{ji}|$, $\underline{\alpha} = \min_{1\le i\le n}\underline{\alpha}_i$, $G_M = \max_{1\le i\le n}G_i$ and $p = \max_{1\le i\le n}p_i$, we consider the above $V(z(t),t)$. Obviously, $V(z(t),t)$ is a positive definite and radially unbounded Lyapunov functional.

Choose $\epsilon > 0$ satisfying the following condition:

$$\frac{\epsilon\epsilon_1}{\underline{\alpha}} - \epsilon_1\gamma + \frac{pG_M\epsilon}{\underline{\alpha}} + rG_M^2\epsilon\tau\,\mathrm{e}^{\epsilon\tau} < 0. \tag{4.5}$$

Since $p_i\int_0^{z_i(t)}(g_i(s)/\alpha_i(s))\,\mathrm{d}s \le p\int_0^{z_i(t)}(G_i s/\underline{\alpha})\,\mathrm{d}s \le (pG_M/2\underline{\alpha})z_i^2(t)$ and $\int_0^{z_i(t)}(s/\alpha_i(s))\,\mathrm{d}s \le \int_0^{z_i(t)}(s/\underline{\alpha})\,\mathrm{d}s \le (1/2\underline{\alpha})z_i^2(t)$, we have

$$\frac{d}{dt}(e^{\epsilon t}V(z(t),t)) \le e^{\epsilon t}\left(\frac{\epsilon\epsilon_1}{\underline{\alpha}}z^{\mathrm{T}}z - \epsilon_1 z^{\mathrm{T}}\Gamma z + \frac{pG_M\epsilon}{\underline{\alpha}}z^{\mathrm{T}}z\right) + \frac{\epsilon e^{\epsilon t}}{1-\eta}\sum_{i=1}^{n}\sum_{j=1}^{n}\int_{t-\tau_{ji}(t)}^{t} r_j|b_{ji}|g_i^2(z_i(v))\,dv \le e^{\epsilon t}$$
$$\times\left(\frac{\epsilon\epsilon_1}{\underline{\alpha}} - \epsilon_1\gamma + \frac{pG_M\epsilon}{\underline{\alpha}}\right)z^{\mathrm{T}}z + \frac{\epsilon e^{\epsilon t}}{1-\eta}\sum_{i=1}^{n}\sum_{j=1}^{n}\int_{t-\tau_{ij}(t)}^{t} r_i|b_{ij}|g_j^2(z_j(v))\,dv. \tag{4.6}$$

Integrating both sides of (4.6) from 0 to an arbitrary positive number s, we can obtain

$$e^{\epsilon s}V(z(s),s) - V(z(0),0) \le \int_0^s e^{\epsilon t}\left(\frac{\epsilon\epsilon_1}{\underline{\alpha}} - \epsilon_1\gamma + \frac{pG_M\epsilon}{\underline{\alpha}}\right)z^{\mathrm{T}}(t)z(t)\,dt$$
$$+\frac{\epsilon}{1-\eta}\int_0^s e^{\epsilon t}\sum_{i=1}^{n}\sum_{j=1}^{n}\int_{t-\tau_{ij}(t)}^{t} r_i|b_{ij}|g_j^2(z_j(v))\,dv\,dt. \tag{4.7}$$

It is analogous to proof of Theorem 3.5 in [25], we can have

$$\epsilon\int_0^s e^{\epsilon t}\int_{t-\tau_{ij}(t)}^{t} r_i|b_{ij}|g_j^2(z_j(v))\,dv\,dt \le \epsilon r_i|b_{ij}\int_0^s e^{\epsilon t}\int_{t-\tau}^{t} g_j^2(z_j(v))\,dv\,dt < r_i|b_{ij}|G_M^2\epsilon\tau e^{\epsilon\tau}$$
$$\times\left(\int_{-\tau}^{0} e^{\epsilon v}z_j^2(v)\,dv + \int_0^s e^{\epsilon v}z_j^2(v)\,dv\right). \tag{4.8}$$

Substituting (4.8) into (4.7) and using (4.5), we can obtain

$$e^{\epsilon s}V(z(s),s) - V(z(0),0) \le \int_0^s e^{\epsilon t}\left(\frac{\epsilon\epsilon_1}{\underline{\alpha}} - \epsilon_1\gamma + \frac{pG_M\epsilon}{\underline{\alpha}}\right)z^{\mathrm{T}}z\,dt + \sum_{i=1}^{n} rG_M^2\epsilon\tau e^{\epsilon\tau}$$
$$\times\left(\int_{-\tau}^{0} e^{\epsilon v}z_i^2(v)\,dv + \int_0^s e^{\epsilon v}z_i^2(v)\,dv\right) \le rG_M^2\epsilon\tau e^{\epsilon\tau}\int_{-\tau}^{0} e^{\epsilon v}z^{\mathrm{T}}(v)z(v)\,dv \equiv M_1\|\phi\|_2^2.$$

So

$$V(z(t),t) \le (V(z(0),0) + M_1\|\phi\|_2^2)e^{-\epsilon t}, \quad \forall t>0. \tag{4.9}$$

$$V(z(0),0) = 2\epsilon_1\sum_{i=1}^{n}\int_0^{z_i(0)}\frac{s}{\alpha_i(s)}\,ds + 2\sum_{i=1}^{n}p_i\int_0^{z_i(0)}\frac{g_i(s)}{\alpha_i(s)}\,ds$$
$$+\frac{1}{1-\eta}\sum_{i=1}^{n}\sum_{j=1}^{n}\int_{-\tau_{ji}(0)}^{0} r_j|b_{ji}|g_i^2(\varphi_i(v))\,dv \le \left(\frac{\epsilon_1}{\underline{\alpha}} + \frac{pG_M}{\underline{\alpha}} + \frac{rG_M^2}{1-\eta}\right)\|\phi\|_2^2 = M_2\|\phi\|_2^2.$$

According to (4.4), (4.9) and the above inequality, we can obtain

$$\frac{\epsilon_1}{\bar{\alpha}}z^{\mathrm{T}}(t)z(t) \le 2\epsilon_1\sum_{i=1}^{n}\int_0^{z_i(t)}\frac{s}{\alpha_i(s)}\,ds \le V(z(t),t) \le (M_1+M_2)\|\phi\|_2^2e^{-\epsilon t}, \quad \forall t>0,$$

where $\bar{\alpha} = \max_{1\le j\le n}\{\bar{\alpha_j}\}$, that is,

$$\|z\|_2 \le \sqrt{\frac{\bar{\alpha}}{\epsilon_1}(M_1+M_2)}\|\phi\|_2e^{-(\epsilon/2)t}. \tag{4.10}$$

(4.10) implies the origin of (4.2) is globally exponentially stable. □

If we set $\tau_{ij}(t) = \tau_{ij}$, i.e., $\dot{\tau}_{ij}(t) = 0$, then we can easily obtain the following result.

Corollary 4.2. *Suppose that in systems (1.2), $\dot{\tau}_{ij}(t) = 0$, $0 \le \tau_{ij} \le \tau$, and* Assumptions A_1– A_3*are satisfied. If there exist symmetric positive diagonal matrices $P = \text{diag}(p_1, p_2, \dots, p_n)$, and $Q = \text{diag}(q_1, 1_2, \dots, q_n)$ such that:*

$$\Omega_1 = 2P\Gamma G^{-1} - PA - A^{\mathrm{T}}P - (PQ^{-1}B)_{\infty} - (PQB)_1 > 0,$$

then the unique equilibrium point x^* of neural system (1.2) is globally exponentially stable.

Corollary 4.3. *Suppose that in systems (1.2), $0 \le \tau_{ij}(t) = \tau_{ij} \le \tau$, and* Assumptions A_1– A_3*are satisfied. If there exist positive constants d_i, $i = 1, 2, \dots, n$, $r_1 \in [0, 1]$, $r_2 \in [0, 1]$, and the following conditions hold:*

$$\max_{1\le i\le n}\left\{\frac{1}{\gamma_i d_i}\left(d_i\sum_{j=1}^{n}(G_i^{2r_1}|a_{ij}| + G_i^{2r_2}|b_{ij}|) + \sum_{j=1}^{n}d_j(G_j^{2(1-r_1)}|a_{ji}| + G_j^{2(1-r_2)}|b_{ji}|)\right)\right\} < 2, \tag{4.11}$$

then the unique equilibrium point x^* of neural system (1.2) is globally exponentially stable.

Proof. Choose $D = \text{diag}(d_1, d_2, \dots, d_n)$, $P = DG$ and $Q = \text{diag}(G_1^{1-2r_2}, G_2^{1-2r_2}, \dots, G_n^{1-2r_2})$, then the Ω_1 in Corollary 4.2 becomes

$$\Omega_1 = 2D\Gamma - DGA - A^{\mathrm{T}}DG - (DGQ^{-1}B)_{\infty} - (DGQB)_1.$$

Thus, for all $x = (x_1, x_2, \dots, x_n) \ne 0$, we have

$$\begin{aligned} x^{\mathrm{T}}\Omega_1 x &= \sum_{i=1}^{n}2d_i\gamma_i x_i^2 - \sum_{i=1}^{n}\sum_{j=1}^{n}2a_{ij}d_iG_ix_ix_j - \sum_{i=1}^{n}\sum_{j=1}^{n}d_iG_i^{2r_2}|b_{ij}|x_i^2 - \sum_{i=1}^{n}\sum_{j=1}^{n}d_jG_j^{2-2r_2}|b_{ji}|x_i^2 \ge \sum_{i=1}^{n}2d_i\gamma_i x_i^2 \\ &\quad -\sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}|d_i(G_i^{2r_1}x_i^2 + G_i^{2(1-r_1)}x_j^2) - \sum_{i=1}^{n}\left(\sum_{j=1}^{n}d_iG_i^{2r_2}|b_{ij}|x_i^2 + \sum_{j=1}^{n}d_jG_j^{2-2r_2}|b_{ji}|x_i^2\right) \\ &= \sum_{i=1}^{n}\left(2d_i\gamma_i - \left(d_i\sum_{j=1}^{n}(G_i^{2r_1}|a_{ij}| + G_i^{2r_2}|b_{ij}|) + \sum_{j=1}^{n}d_j(G_j^{2(1-r_1)}|a_{ji}| + G_j^{2(1-r_2)}|b_{ji}|)\right)\right)x_i^2 > 0. \end{aligned}$$

Therefore, The Corollary then follows from Corollary 4.2. □

If we choose the matrices $P = I$, $Q = qI$, as a special cases of Theorem 4.1, we have the following corollary.

Corollary 4.4. *Suppose that in systems (1.2), $\dot{\tau}_{ij}(t) \le \eta < 1$for non-negative constant η (If $\dot{\tau}_{ij}(t) \le \eta \le 0$, let $\eta = 0$), $0 \le \tau_{ij}(t) = \tau_{ij} \le \tau$, and* Assumptions A_1– A_3 *are satisfied. If*

$$\Omega_{12} = 2\Gamma G^{-1} - A - A^{\mathrm{T}} - \frac{1}{q(1-\eta)}\|B\|_{\infty} - q\|B\| > 0,$$

then the unique equilibrium point x^* of neural system (1.2) is globally exponentially stable.

By constructing a differential Lyapunov functional, we have the following results

Theorem 4.5. *Suppose that in systems (4.2), $\dot{\tau}_{ij}(t) \le \eta < 1$for non-negative constant η* (If $\dot{\tau}_{ij}(t) \le \eta \le 0$, *let* $\eta = 0$), $0 \le \tau(t) \le \tau$, Assumptions A_1– A_3 *are satisfied. The origin of neural system (4.2) is globally exponentially stable if there exist symmetric positive diagonal matrices $P = diag(p_1, p_2, \dots, p_n)$ and $Q = \text{diag}(q_1, q_2, \dots, q_n)$ such that:*

$$\Omega_{22} = 2P\Gamma G^{-1} - PA - A^{\mathrm{T}}P - \frac{1}{1-\eta}(PBQ^{-1})_{\infty} - (PBQ)_1 > 0.$$

Proof. We employ the following positive-definite Lyapunov functional:

$$V(z(t), t) = \epsilon_1 V_3(z(t), t) + V_4(z(t), t), \tag{4.12}$$

where

$$V_3(z(t),t)=2\sum_{i=1}^{n}p_i\int_0^{z_i(t)}\frac{s}{\alpha_i(s)}\,\mathrm{d}s,$$

$$V_4(z(t),t)=2\sum_{i=1}^{n}p_i\int_0^{z_i(t)}\frac{g_i(s)}{\alpha_i(s)}\,\mathrm{d}s+\frac{1}{1-\eta}\sum_{i=1}^{n}\sum_{j=1}^{n}\int_{t-\tau_{ij}(t)}^{t}p_ir_j|b_{ij}|g_j^2(z_j(v))\,\mathrm{d}v,$$

for some positive constants ϵ_1 and $r_k(k=1,2,\ldots,m)$. The positive constants ϵ_1 and $r_k(k=1,2,\ldots,m)$ will be determined later. It is analogous to that of Theorem 4.1, we can obtain

$$\dot V_3(z(t),t)\le -z^{\mathrm{T}}(t)P\Gamma z(t)+Mg(z(t))^{\mathrm{T}}g(z(t))+M\sum_{i=1}^{n}\sum_{j=1}^{n}p_i|b_{ij}|g_j^2(z_j(t-\tau_{ij}(t)))$$

where $M=\max\{2\|A^{\mathrm{T}}P\Gamma^{-1}A\|_2,\max_{1\le i,j\le n}(2/\gamma_i)|b_{ij}|\}\ge 0$.

By using the inequality $2g_i(z_i(t))b_{ij}g_j(z_j(t-\tau_{ij}(t)))\le q_j^{-1}|b_{ij}|g_i^2(z_i(t))+q_j|b_{ij}|g_j^2(z_j(t-\tau_{ij}))$, we also have

$$\dot V_4(z(t),t)\le -g(z(t))^{\mathrm{T}}(2P\Gamma G^{-1}-PA-A^{\mathrm{T}}P-(PBQ^{-1})_\infty)g(z(t))+\sum_{i=1}^{n}\sum_{j=1}^{n}p_jr_i|b_{ji}|g_i^2(z_i(t))$$
$$-\sum_{i=1}^{n}\sum_{j=1}^{n}(r_j-q_j)p_i|b_{ij}|g_j^2(z_j(t-\tau_{ij}(t))).$$

Since $\Omega_{22}>0$, there exists $\epsilon_2>0$ such that $\Omega_{22}-\epsilon_2(I+(1/(1-\eta))B_1)>0$. If we define $R=\mathrm{diag}(r_1,r_2,\ldots,r_n)=Q+\epsilon_2 I$, we can have

$$\dot V_4(z(t),t)\le -\epsilon_2 g(z(t))^{\mathrm{T}}g(z(t))-\epsilon_2\sum_{i=1}^{n}\sum_{j=1}^{n}p_i|b_{ij}|g_j^2(z_j(t-\tau_{ij}(t)))\,\mathrm{d}s.$$

If we choose $\epsilon_1>0$ such that $M\epsilon_1\le\epsilon_2$, then $\dot V(z(t),t)\le -\epsilon_1 z^{\mathrm{T}}(t)P\Gamma z(t)$.

The rest of proof is analogous to that of Theorem 4.1 and hence, is omitted here. □

By choosing $V_4'(z(t),t)=2\sum_{i=1}^{n}p_i\int_0^{z_i(t)}(g_i(s)/\alpha_i(s))\,\mathrm{d}s+(1/(1-\eta))\sum_{i=1}^{n}\sum_{j=1}^{n}\int_{t-\tau_{ij}(t)}^{t}p_ir_jb_{ij}^2g_j^2(z_j(v))\,\mathrm{d}v$, we can obtain

Theorem 4.6. *Suppose that in systems (4.2), $\dot\tau_{ij}(t)\le\eta<1$for non-negative constant η (If $\dot\tau_{ij}(t)\le\eta\le 0$, let $\eta=0$), $0\le\tau(t)\le\tau$,* Assumptions *A_1– A_3 are satisfied. The origin of neural system (4.2) is globally exponentially stable if there exist symmetric positive diagonal matrices $P=diag(p_1,p_2,\ldots,p_n)$ and $Q=diag(q_1,q_2,\ldots,q_n)$ such that:*

$$\Omega_{33}=2P\Gamma G^{-1}-PA-A^{\mathrm{T}}P-\frac{1}{1-\eta}PQdiag\left(\sum_{j=1}^{n}b_{j1}^2,\sum_{j=1}^{n}b_{j2}^2,\ldots,\sum_{j=1}^{n}b_{jn}^2\right)-nPQ^{-1}>0.$$

5. The Cohen–Grossberg model with distributed delays

In this section, we consider system (1.3) with initial values of the form

$$x_i(s)=\varphi_i(s),\quad s\in(-\infty,0],\quad i=1,2,\ldots,n,\tag{5.1}$$

where $\varphi_i(s)$ denote real-valued continuous and bounded functions defined on $(-\infty,0]$.

Firstly, about the kernel functions $k_{ij}(t)$, $i,j=1,2,\ldots,n$, we also have the following lemma due to [24]:

Lemma 5.1. *If* Assumption *A_4holds, then*

$$\int_0^\infty sk_{ij}(s)\,\mathrm{d}s<\infty,\ \int_0^\infty k_{ij}(s)se^{\epsilon s}\,\mathrm{d}s<\infty,\quad \forall 0<\epsilon<\frac{\delta_0}{2},\quad i,j=1,2,\ldots,n.\tag{5.2}$$

By means of the transformation (4.3), (1.3) can easily be reduced to

$$\dot{z}_i(t) = -\alpha_i(z_i(t))\left[\beta_i(z_i(t)) - \sum_{j=1}^{n} a_{ij} g_j(z_j(t)) - \sum_{j=1}^{n} b_{ij}\int_{-\infty}^{t} k_{ij}(t-s) g_j(z_j(s))\,\mathrm{d}s\right]. \tag{5.3}$$

Note that (1.2) and (1.3) have the same equilibrium set. Thus, if all the conditions in Theorem 3.3 are satisfied, then (1.3) also has an equilibrium x^*. Obviously, the stability property of x^* for neural system (1.3) is the same as the stability property of the origin for neural system (5.3). Thus in the following, we only consider the stability property of the origin of (5.3). We now present our main global exponential stability results.

Theorem 5.2. *Suppose that in systems* (5.3), Assumptions A_1– A_3*are satisfied. The origin of neural system* (5.3) *is globally asymptotically stable if there exist symmetric positive diagonal matrices* $P = \mathrm{diag}(p_1, p_2, \ldots, p_n)$ *and* $Q = \mathrm{diag}(q_1, q_2, \ldots, q_n)$ *such that:*

$$\Omega_1 = 2P\Gamma G^{-1} - PA - A^{\mathrm{T}}P - (PQ^{-1}B)_\infty - (PQB)_1 > 0.$$

In addition, Assumption A_4 *is satisfied, then the origin of system* (5.3) *is globally exponentially stable. This implies that there exist positive constants k and* γ *such that for any solution* $x(t)$ *of system* (5.3) *with initial function* $x(s) = \phi(s)$ *for all* $s \in (-\infty, 0]$, *where* $\phi \in C((-\infty, 0], \mathbb{R}^n)$, *one has*

$$\sum_{i=1}^{n} z_i^2(t) \le \gamma \mathrm{e}^{-kt} \sum_{i=1}^{n} \sup_{s\in(-\infty,0]} \phi_i^2(s) = \gamma \mathrm{e}^{-kt}\|\phi\|_2.$$

Proof. We employ the following positive-definite Lyapunov functional:

$$V(z(t), t) = \epsilon_1 V_1(z(t), t) + V_5(z(t), t), \tag{5.4}$$

where

$$V_5(z(t), t) = 2\sum_{i=1}^{n} p_i \int_0^{z_i(t)} \frac{g_i(s)}{\alpha_i(s)}\,\mathrm{d}s + \sum_{i=1}^{n}\sum_{j=1}^{n}\int_0^{+\infty} r_j|b_{ji}|k_{ji}(s)\int_{t-s}^{t} g_i^2(z_i(v))\,\mathrm{d}v\,\mathrm{d}s,$$

for some positive constants ϵ_1 and $r_k (k = 1, 2, \ldots, m)$. The positive constants ϵ_1 and $r_k (k = 1, 2, \ldots, m)$ will be determined later.

It is analogous to that of Theorem 4.1, we can choose $\epsilon_2 > 0$ such that the matrix $2\Gamma - \epsilon_2 AA^{\mathrm{T}} - \epsilon_2 B_\infty$ is a positive definite matrix. By Lemma 3.2, we gain

$$\dot{V}_1(z(t), t) \le -z^{\mathrm{T}}(t)(2\Gamma - \epsilon_2 AA^{\mathrm{T}} - \epsilon_2 B_\infty)z(t) + \frac{1}{\epsilon_2} g(z(t))^{\mathrm{T}} g(z(t)) + \frac{1}{\epsilon_2}\sum_{i=1}^{n}\sum_{j=1}^{n}\int_0^{+\infty} |b_{ij}|k_{ij}(s) g_j^2(z_j(t-s))\,\mathrm{d}s.$$

For the term $\dot{V}_5$, we can also have

$$\dot{V}_5(z(t), t) \le -g(z(t))^{\mathrm{T}}(2P\Gamma G^{-1} - PA - A^{\mathrm{T}}P - (PQ^{-1}B)_\infty) g(z(t)) + \sum_{i=1}^{n}\sum_{j=1}^{n} r_j|b_{ji}| g_i^2(z_i(t))$$
$$-\sum_{i=1}^{n}\sum_{j=1}^{n}(r_i - p_i q_i)|b_{ij}|\int_0^{\infty} k_{ij}(s) g_j^2(z_j(t-s))\,\mathrm{d}s.$$

Since $\Omega_1 > 0$, there exists $\epsilon_3 > 0$ such that $\Omega_1 - \epsilon_3(I + B_1) > 0$. If we define $R = \mathrm{diag}(r_1, r_2, \ldots, r_n) = PQ + \epsilon_3 I$ which is a symmetric positive diagonal matrix, we can have

$$\dot{V}_5(z(t),t) \le -g(z(t))^{\mathrm{T}}(\Omega_3 - \epsilon_3 B_1)g(z(t)) - \epsilon_3 \sum_{i=1}^{n}\sum_{j=1}^{n}|b_{ij}| \int_0^{\infty} k_{ij}(s) g_j^2(z_j(t-s))\,\mathrm{d}s \le -g(z(t))^{\mathrm{T}}(\Omega_1$$
$$-\epsilon_3(I+B_1)+\epsilon_3 I)g(z(t)) - \epsilon_3 \sum_{i=1}^{n}\sum_{j=1}^{n}|b_{ij}| \int_0^{\infty} k_{ij}(s) g_j^2(z_j(t-s))\,\mathrm{d}s \le -\epsilon_3 g(z(t))^{\mathrm{T}} g(z(t))$$
$$-\epsilon_3 \sum_{i=1}^{n}\sum_{j=1}^{n}|b_{ij}| \int_0^{\infty} k_{ij}(s) g_j^2(z_j(t-s))\,\mathrm{d}s.$$

If we choose $\epsilon_1 > 0$ such that $\epsilon_1 \le \epsilon_2\epsilon_3$, then $\dot{V}(z(t)) \le -\epsilon_1 z(t)^{\mathrm{T}}(2\Gamma - \epsilon_2 AA^{\mathrm{T}} - \epsilon_2 B_\infty)z(t)$.

On the other hand, obviously,

$$\frac{\epsilon_1}{\bar{a}}|z(t)|^2 = \frac{\epsilon_1}{\bar{a}} z^{\mathrm{T}}(t)z(t) \le 2\epsilon_1 \sum_{i=1}^{n} \int_0^{z_i(t)} \frac{s}{\alpha_i(s)}\,\mathrm{d}s \le V(z(t),t).$$

Thus, the origin of neural system (5.3) is globally asymptotically stable.

If Assumption A_4 is satisfied. Let $\gamma = \lambda_{\min}(2\Gamma - \epsilon_2 AA^{\mathrm{T}} - \epsilon_2 B_\infty)$, $r = \max_{1\le k\le n}|r_k|$, $\underline{\alpha} = \min_{1\le i\le n}\underline{\alpha}_i$, $G_M = \max_{1\le i\le n} G_i$ and $p = \max_{1\le i\le n} p_i$, we consider the above $V(z(t),t)$. Obviously, $V(z(t),t)$ is a positive definite and radially unbounded Lyapunov functional.

By virtue of (5.1), it follows that $k_{ij} \equiv \int_0^{+\infty} k_{ij}(t)t\mathrm{e}^{\epsilon t}\,\mathrm{d}t < \infty, \forall \epsilon \in (0,(\delta_0/2))$.

Let $k = \max_{1\le i\le n} k_{ij}$, choose $\epsilon \in (0,(\delta_0/2))$ satisfying the following condition:

$$\frac{\epsilon\epsilon_1}{\underline{\alpha}} - \epsilon_1\gamma + \frac{pG_M\epsilon}{\underline{\alpha}} + \|RB\|_1 k G_M^2 \epsilon < 0. \tag{5.5}$$

It is analogous to that of Theorem 4.1, we then have

$$\frac{d}{\mathrm{d}t}(\mathrm{e}^{\epsilon t}V(z(t),t)) = \epsilon \mathrm{e}^{\epsilon t} V(z(t),t) + \mathrm{e}^{\epsilon t}\frac{\mathrm{d}v(z(t),t)}{\mathrm{d}t} \le \mathrm{e}^{\epsilon t}\left(\frac{\epsilon\epsilon_1}{\underline{\alpha}} - \epsilon_1\gamma + \frac{pG_M\epsilon}{\underline{\alpha}}\right) z^{\mathrm{T}}(t)z(t)$$
$$+\epsilon \mathrm{e}^{\epsilon t}\sum_{i=1}^{n}\sum_{j=1}^{n}\int_0^{+\infty} r_j |b_{ji}| k_{ji}(s) \int_{t-s}^{t} g_i^2(z_i(v))\,\mathrm{d}v\,\mathrm{d}s. \tag{5.6}$$

by integrating both sides of (5.6) from 0 to an arbitrary positive number θ and changing the integrals, it is analogous to proof of Theorem 1 in [24] we have

$$\mathrm{e}^{\epsilon\theta} V(z(\theta),\theta) - V(z(0),0) \le \|RB\|_1 k G_M^2 \epsilon \int_{-\infty}^{0} \mathrm{e}^{\epsilon v} z^{\mathrm{T}}(v)z(v)\,\mathrm{d}v \equiv M_1 \|\phi\|_\infty^2.$$

So

$$V(z(t),t) \le (V(z(0),0) + M_1\|\phi\|_\infty^2)\mathrm{e}^{-\epsilon t}, \quad \forall t > 0. \tag{5.7}$$

$$V(z(0),0) = 2\epsilon_1 \sum_{i=1}^{n}\int_0^{z_i(0)} \frac{s}{\alpha_i(s)}\,\mathrm{d}s + 2\sum_{i=1}^{n} p_i \int_0^{z_i(0)} \frac{g_i(s)}{\alpha_i(s)}\,\mathrm{d}s + \sum_{i=1}^{n}\sum_{j=1}^{n}\int_{-\infty}^{0} r_i |b_{ji}| k_{ji}(s)$$
$$\times \int_{t-s}^{t} g_i^2(z_i(v))\,\mathrm{d}v\,\mathrm{d}s \le \left(\frac{\epsilon_1}{\underline{\alpha}} + \frac{pG_M}{\underline{\alpha}} + k_{\mathrm{s}}\|RB\|_1 G_M^2\right)\|\phi\|_\infty^2 = M_2\|\phi\|_\infty^2,$$

where $k_s = \max_{1\le i\le n, 1\le j\le n}\{\int_0^{+\infty} s k_{ji}(s)\,\mathrm{d}s\}$. According to (5.4), (5.7) and the above inequality, we can obtain

$$\frac{\epsilon_1}{\bar{\alpha}} z^{\mathrm{T}}(t)z(t) \le 2\epsilon_1 \sum_{i=1}^{n}\int_0^{z_i(t)} \frac{s}{\alpha_i(s)}\mathrm{d}s \le V(z(t),t) \le (M_1+M_2)\|\phi\|_\infty^2 \mathrm{e}^{-\epsilon t}, \quad \forall t > 0,$$

where $\bar{\alpha} = \max_{1\le j\le n}\{\bar{\alpha}_j\}$, that is

$$\|z\| \le \sqrt{\frac{\bar{\alpha}(M_1+M_2)}{\epsilon_1}}\|\phi\|_\infty e^{-(\epsilon/2)t}. \tag{5.8}$$

(5.8) implies the origin of (5.3) is globally exponentially stable. □

Corollary 5.3. *Suppose that in systems* (1.3), Assumptions A_1– A_3*are satisfied. If there exist positive constants* $d_i, i = 1, 2, \ldots, n, r_1 \in [0, 1], r_2 \in [0, 1]$, *and the condition* (4.11) *holds, then the unique equilibrium point* x^* *of neural system* (1.3) *is globally exponentially stable*.

Proof. Choose $D = \mathrm{diag}(d_1, d_2, \ldots, d_n)$, $P = DG$ and $Q = \mathrm{diag}(G_1^{1-2r_2}, G_2^{1-2r_2}, \ldots, G_n^{1-2r_2})$, then the Ω_1 in Theorem 4.1 becomes

$$\Omega_1 = 2D\Gamma - DGA - A^{\mathrm{T}}DG - (DGQ^{-1}B)_\infty - (DGQB)_1.$$

By Corollary 4.3, we have $\Omega_1 > 0$.Thus, The Corollary then follows from Theorem 5.2. □

By constructing a differential Lyapunov functional, we have the following results

Theorem 5.4. *Suppose that in systems* (5.3), Assumptions A_1– A_3*are satisfied. The origin of neural system* (5.3) *is globally asymptotically stable if there exist symmetric positive diagonal matrices* $P = \mathrm{diag}(p_1, p_2, \ldots, p_n)$ *and* $Q = diag(q_1, q_2, \ldots, q_n)$ *such that:*

$$\Omega_2 = 2P\Gamma G^{-1} - PA - A^{\mathrm{T}}P - (PBQ^{-1})_\infty - (PBQ)_1 > 0.$$

In addition, Assumption A_4 is satisfied, then the origin of system (5.3) is globally exponentially stable.

Proof. We employ the following positive-definite Lyapunov functional:

$$V(z(t), t) = \epsilon_1 V_1(z(t), t) + V_6(z(t), t), \tag{5.9}$$

where

$$V_6(z(t), t) = 2\sum_{i=1}^n p_i \int_0^{z_i(t)} \frac{g_i(s)}{\alpha_i(s)}\,\mathrm{d}s + \sum_{i=1}^n\sum_{j=1}^n \int_0^{+\infty} p_i r_j |b_{ij}| k_{ij}(s) \int_{t-s}^t g_j^2(z_j(v))\,\mathrm{d}v\,\mathrm{d}s,$$

for some positive constants ϵ_1 and $r_k (k = 1, 2, \ldots, m)$. The positive constants ϵ_1 and $r_k (k = 1, 2, \ldots, m)$ will be determined later.

It is similar to that of Theorem 5.2, we can obtain that there exist $\epsilon_2 > 0$ such that the matrix $2\Gamma - \epsilon_2 AA^{\mathrm{T}} - \epsilon_2 (PB)_\infty$ is a positive definite matrix and

$$\dot{V}_1(z(t), t) \le -z^{\mathrm{T}}(t)(2\Gamma - \epsilon_2 AA^{\mathrm{T}} - \epsilon_2(PB)_\infty)z(t) + \frac{1}{\epsilon_2} g(z(t))^{\mathrm{T}} g(z(t))$$
$$+\frac{1}{\epsilon_2}\sum_{i=1}^n\sum_{j=1}^n \int_0^{+\infty} p_i |b_{ij}| k_{ij}(s) g_j^2(z_j(t-s))\,\mathrm{d}s.$$

By using the inequality $2g_i(z_i(t))b_{ij}k_{ij}(s)g_j(z_j(t-s)) \le q_j^{-1}|b_{ij}|k_{ij}(s)g_i^2(z_i(t)) + q_j|b_{ij}|k_{ij}(s)g_j^2(z_j(t-s))$, we also have

$$\dot{V}_6(z(t), t) \le -g(z(t))^{\mathrm{T}}(2P\Gamma G^{-1} - PA - A^{\mathrm{T}}P - (PBQ^{-1})_\infty)g(z(t)) + \sum_{i=1}^n\sum_{j=1}^n p_j r_i |b_{ji}| g_i^2(z_i(t))$$
$$-\sum_{i=1}^n\sum_{j=1}^n (r_j - q_j) p_i |b_{ij}| \int_0^\infty k_{ij}(s) g_j^2(z_j(t-s))\,\mathrm{d}s.$$

Since $\Omega_2 > 0$, there exists $\epsilon_3 > 0$ such that $\Omega_2 - \epsilon_3(I + (PB)_1) > 0$. If we define $R = \text{diag}(r_1, r_2, \ldots, r_n) = Q + \epsilon_3 I$, we can have

$$\dot{V}_6(z(t), t) \leq -\epsilon_3 g(z(t))^{\mathrm{T}} g(z(t)) - \epsilon_3 \sum_{i=1}^{n} \sum_{j=1}^{n} p_i |b_{ij}| \int_0^{\infty} k_{ij}(s) g_j^2(z_j(t-s))\, \mathrm{d}s.$$

If we choose $\epsilon_1 > 0$ such that $\epsilon_1 \leq \epsilon_2 \epsilon_3$, then $\dot{V}(z(t), t) \leq -\epsilon_1 z(t)^{\mathrm{T}}(2\Gamma - \epsilon_2 A A^{\mathrm{T}} - \epsilon_2 (PB)_{\infty}) z(t)$.

The rest of proof is similar to that of Theorem 5.2 and hence, is omitted here. □

By choosing $V_2(z(t), t) = 2\sum_{i=1}^{n} p_i \int_0^{z_i(t)} \frac{g_i(s)}{\alpha_i(s)}\, \mathrm{d}s + \sum_{i=1}^{n}\sum_{j=1}^{n} \int_0^{\infty} \int_{t-s}^{t} p_i r_j b_{ij}^2 k_{ij}(s) g_j^2(z_j(v))\, \mathrm{d}v\, \mathrm{d}s$, we can obtain.

Theorem 5.5. *Suppose that in systems (5.3),* Assumptions A_1– A_4*are satisfied. The origin of neural system (5.3) is globally exponentially stable if there exist symmetric positive diagonal matrices* $P = \text{diag}(p_1, p_2, \ldots, p_n)$ *and* $Q = \text{diag}(q_1, q_2, \ldots, q_n)$ *such that:*

$$\Omega_3 = 2P\Gamma G^{-1} - PA - A^{\mathrm{T}} P - nPQ^{-1} - PQ\text{diag}\left(\sum_{j=1}^{n} b_{j1}^2, \sum_{j=1}^{n} b_{j2}^2, \ldots, \sum_{j=1}^{n} b_{jn}^2\right) > 0.$$

Choosing the matrices $P = I$, $Q = I$, as a special cases of Theorem 5.4, we have the following corollary.

Corollary 5.6. *Suppose that in systems (1.3),* Assumptions A_1– A_4*are satisfied. If*

$$2\Gamma G^{-1} - A - A^{\mathrm{T}} - B_{\infty} - B_1 > 0,$$

then the unique equilibrium x^* *of neural system (1.3) is globally exponentially stable.*

Choosing the matrices $P = I$, $Q = G$, as a special cases of Theorem 5.4, we also obtain the following corollary.

Corollary 5.7. *Suppose that in systems (1.3),* Assumptions A_1– A_4*are satisfied. If*

$$2\Gamma G^{-1} - A - A^{\mathrm{T}} - (BG^{-1})_{\infty} - (BG)_1 > 0,$$

then the unique equilibrium x^* *of neural system (1.3) is globally exponentially stable.*

6. Remark and numerical example

Several results on the globally exponential stability of neural networks with delays have appeared in recent years. First, comparing with the corresponding results given in [2,6,11,12,15,23–25], we find that the results obtained in this paper very strongly improve and extend those results in many aspects. In [2,12,15], the authors gave some global asymptotical (exponential) stability criteria for a class of Cohen–Grossberg neural networks with bounded activation functions and fixed delays. However, in this paper, we do not require that the activation functions to be bounded and delays are constants. In additions, the authors in [6] also removed the boundedness of activation functions, and considered Hopfield neural networks with single delay $\tau(t)$. But, the authors have not proved the existence of equilibrium point. However, in this paper, we do not assume that all delays are equal. Moreover, we have proved the existence of equilibrium point. Third, all conditions in [11] and [23] neglect the signs of entries in the connection matrix A, and thus, the difference between excitatory and inhibitory effects might be ignored. Finally, in [24] and [25], the authors have only considered the case when $\tau_{ij}(t) = \tau_j(t)$. Therefore, our results in this paper extend and improve the corresponding results in [2,6,11,12,15,23–25]. Second, we will now compare our results with those results in [11,23]. First, we restate the previous stability results:

Theorem 6.1. [11] *For delayed neural networks (1.2), suppose that* Assumptions A_1– A_3*are satisfied. System (1.2) is globally exponentially stable If there exists constants* $h_{ij}, l_{ij}, h^*_{ij}, l^*_{ij} \in R$, $\omega_i > 0 (i, j = 1, 2, \ldots, n)$, $r > 1$ *and* $\sigma > 0$ *such that*

$$\xi_i \equiv r\omega_i \underline{\alpha}_i \gamma_i - (r-1)\sum_{j=1}^{n} \omega_j \bar{\alpha}_i |a_{ij}|^{((r-h_{ij})/r-1)} G_j^{((r-l_{ij})/r-1)} - \sum_{j=1}^{n} \omega_j \bar{\alpha}_i G_j^{l_{ij}} |a_{ij}|^{h_{ij}}$$
$$-(r-1)\sum_{j=1}^{n} \omega_j \bar{\alpha}_i |b_{ij}|^{((r-h^*_{ij})/r-1)} G_j^{((r-l^*_{ij})/r-1)} - \sum_{j=1}^{n} \omega_j \bar{\alpha}_i G_j^{l^*_{ij}} |b^*_{ij}|^{h_{ij}} > \sigma, i = 1, 2, \ldots, n.$$

Theorem 6.2. [23] *Suppose that in systems (1.3),* $a_i(x) = 1$, $b_i(x) = d_i x$, Assumptions A_1– A_4*are satisfied. If there exist positive constants* d_i, $i = 1, 2, \ldots, n$, $r_1 \in [0, 1]$, $r_2 \in [0, 1]$, *and the following condition holds:*

$$\rho(M) < 1, M = (m_{ij})_{n\times n}, m_{ij} = \frac{G_j(|a_{ij}| + |b_{ij}|)}{d_i} \tag{6.1}$$

where $\rho(M)$ *denotes the spectral radius of a square matrix M, then the equilibrium* x^* ofbi neural system *(1.3) is globally asymptotically stable*.

Now, we present some example to illustrate the effectiveness of Corollary 4.4 and 5.6

Example 6.3. Consider the following Cohen–Grossberg neural networks with discrete delays:

$$\begin{aligned}
\frac{dx_1(t)}{dt} &= -(8+\sin(x_1(t)))\left[\gamma x_1(t) + f(x_1(t)) + f(x_2(t)) - f\left(x_1\left(t - \frac{1}{3}\sin t - 1\right)\right)\right.\\
&\quad \left. - f\left(x_2\left(t - \frac{1}{4}e^{-\sin t} - 1\right)\right) + 2\right],\\
\frac{dx_2(t)}{dt} &= -(5+\cos(x_2(t)))\left[\gamma x_2(t) - f(x_1(t)) - f(x_2(t)) - f\left(x_1\left(t - \frac{1}{4}e^{-\sin t} - 1\right)\right)\right.\\
&\quad \left. - f\left(x_2\left(t - \frac{1}{3}\sin t - 1\right)\right) + 3\right],
\end{aligned} \tag{6.2}$$

where $f(x) = (1/2)x + (1/2)\tanh(x)$. In this case, $\underline{\alpha}_1 = 7$, $\underline{\alpha}_2 = 4$, $\bar{\alpha}_1 = 9$, $\bar{\alpha}_2 = 6$, $\tau_{11}(t) = \tau_{22}(t) = (1/3)\sin t + 1$, $\tau_{12}(t) = \tau_{21}(t) = (1/4)e^{-\sin t} + 1$, $\Gamma = \gamma I$, $G = I$, $A = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $\eta = (1/3)$. Thus the Ω_{12} in Corollary 4.4 becomes:

$$\Omega_{12} = \begin{bmatrix} 2\gamma + 2 - \dfrac{3}{q} - 2q & 0 \\ 0 & 2\gamma + 2 - \dfrac{3}{q} - 2q \end{bmatrix}.$$

If $\gamma > \sqrt{6} - 1$, there always exists positive constant q such that $2\gamma + 2 - (3/q) - 2q > 0$. Hence, the conditions of Corollary 4.4 hold for $\gamma > \sqrt{6} - 1$.

Now let us check the conditions of Theorem 6.1 for the same network parameters. In this case, for all $h_{ij}, l_{ij}, h^*_{ij}, l^*_{ij} \in R$, $\omega_i > 0 (i, j = 1, 2)$, $r > 1$, we have

$$\xi_1 = (7\gamma - 18)r\omega_1 - 18r\omega_2, \xi_2 = (4\gamma - 12)r\omega_2 - 12r\omega_1 > 0.$$

In order to ensure the positive definiteness of ξ_1 and ξ_2, γ must be chosen as $\gamma > 3$. Therefore, it can be concluded that if $\sqrt{6} - 1 < \gamma \leq 3$, the conditions of Theorem 6.1 are not satisfied, whereas the conditions of Corollary 4.4 still hold for $\sqrt{6} - 1 < \gamma \leq 3$.

Example 6.4. Consider the following Cohen–Grossberg neural networks with delays:

$$\begin{aligned} \frac{dx_1(t)}{dt} &= -\gamma x_1(t) - f(x_1(t)) - f(x_2(t)) + \int_{-\infty}^{t} 2e^{-2(t-s)} f(x_1(s))\,ds + \int_{-\infty}^{t} e^{-(t-s)} f(x_2(s))\,ds + 2\Bigg], \\ \frac{dx_2(t)}{dt} &= -\gamma x_2(t) + f(x_1(t)) - f(x_2(t)) - \int_{-\infty}^{t} e^{-(t-s)} f(x_1(s))\,ds - \int_{-\infty}^{t} 2e^{-2(t-s)} f(x_2(s))\,ds + 3\Bigg], \end{aligned} \tag{6.3}$$

where $f(x) = (1/2)x + (1/2)\tanh(x)$. In this case, $k_{11}(t) = k_{22}(t) = 2e^{-2t}$, $k_{12}(t) = k_{21}(t) = e^{-t}$, $\Gamma = \gamma I$, $G = I$, $A = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$. Thus the condition in Corollary 5.6 becomes:

$$2\Gamma G^{-1} - A - A^{T} - B_{\infty} - B_1 = \begin{bmatrix} 2\gamma - 2 & 0 \\ 0 & 2\gamma - 2 \end{bmatrix} > 0,$$

Thus, the condition of Corollary 5.6 holds for $\gamma > 1$.

Now let us check the conditions of Theorem 6.2 for the same network parameters. In this case, we have

$$M = \begin{bmatrix} \frac{2}{\gamma} & \frac{2}{\gamma} \\ \frac{2}{\gamma} & \frac{2}{\gamma} \end{bmatrix}, \rho(M) = \frac{4}{\gamma}$$

In order to ensure that $\rho(M) < 1$, γ must be chosen as $\gamma > 4$. Therefore, it can be concluded that if $1 < \gamma \le 4$, the conditions of Theorem 6.2 are not satisfied, whereas the conditions of Corollary 5.6 still hold for $1 < \gamma \le 4$.

7. Conclusion

In this paper, some criteria for the global exponential stability of a class of Cohen–Grossberg neural networks with discrete delays and with distributed delays have been derived. We have also shown by analyses that the neuronal input–output activation function only needs to satisfy Assumption A_3 given in this paper, but does not need to be continuous, differentiable, strictly monotonically increasing, and bounded, as usually required by other analyzing methods. The criteria concerning the differences between excitatory and inhibitory effects on units extend some existing results in the literature. Some new stability conditions are stated in simple algebraic forms so that their verification and applications are straightforward and convenient. Comparisons between our results and the previous results have also been made. They shows that our results establish a new set of globally exponential stability criteria for delayed neural networks. Those conditions are less restrictive than those given in the earlier references.

Acknowledgements

The authors would like to thank the anonymous reviewers and the editor for their constructive comments that led to a truly significant improvement in the manuscript. This work was supported by the Natural Science Foundation of China under Grant 60602038 and China Postdoctoral Science Foundation under Grant 20060400239.

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